3. V. A. Belavin, V. V. Golub, I. M. Naboko, and A. I. Opara, "Investigation of the unsteady flow structure of a shock-heated gas get," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1973).
4. V. V. Volchkov, A. V. Ivanov, N. I. Kislyakov, et al., "Low-density jets expanding from a sonic nozzle with large pressure differences," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1973).
5. V. N. Gusev, "Supersonic nozzle startup," Inzh. Zh., No. 1 (1961).
6. G. A. Simons, "The large-time behavior of a steady spherical source expanding into an arbitrary ambient gas," AIAA Paper No. 70, New York (1970).
7. S. F. Chekmarev, "Unsteady radial expansion of gas into a fluid-filled space from a suddenly switched-on steady source," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1975).
8. S. F. Chekmarev and N. V. Stankus, "Gasdynamic model and similarity relation for the startup of supersonic nozzles and jets," Zh. Tekh. Fiz., 54, No. 8 (1984).
9. V. G. Dulov and G. A. Luk'yanov, "Gasdynamics of Outflow Processes [in Russian], Nauka, Novosibirsk (1984).
10. V. P. Korobeinikov, N. S. Mel'nikova, and E. V. Ryazanov, "Theory of Point Explosion [in Russian], Fizmatgiz, Moscow (1961).
11. V. V. Golub and A. M. Shul'meister, "Starting shock waves and vortex structures associated with jet formation," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1988).

SHOCK WAVES IN POLYDISPERSE BUBBLY MEDIA WITH DISSIPATION

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The shock wave structure in a bubbly liquid with a discrete bubble size distribution function (at each point in space there are bubbles of M different radii) is investigated. In the coordinate system moving with the wave the equations of motion reduce to a dynamical system in 2M-dimensional phase space. For an arbitrary finite M the existence, uniqueness and stability of the corresponding structure are proved. Stability is understood in the sense of satisfaction of the Cemplen theorem at the shock wave, treated within the framework of the equilibrium model as a strong discontinuity.

1. Mathematical Model

For low bubble concentrations the equations of motion of a polydisperse bubbly medium with a incompressible carrier phase have the form [1, 2]:

$$v_t - u_q = 0; \tag{1.1}$$

$$u_t + p_q = 0; \tag{1.2}$$

$$R_i R_{itt} + 3R_{it}^2/2 = \left(p_2^i(R_i) - p\right)/\rho_l - 4\mu_i R_{it}/(\rho_l R_i); \tag{1.3}$$

$$c_{2it} = 0,\ n_{it} = 0,\ i = 1, \ldots, M. \tag{1.4}$$

Here, t is time, q is the mass Lagrangian coordinate, u is velocity, v is the specific volume of the mixture, p is the pressure in the liquid, R_i are the radii of the bubbles, i denotes the corresponding bubble fraction (kind), $p_2^i(R_i)$ is the pressure in a bubble of the i-th kind, ρ_ℓ is the density of the liquid (constant), μ_i are the effective dynamic viscosity coefficients [3, Part 1, pp. 125, 126], c_{2i} are the mass bubble concentrations, and n_i is the number of bubbles per unit mass of mixture. As shown in [3], in liquids with vicosities of the same order as the viscosity of water the damping of fairly large bubbles is mainly determined by thermal dissipation. Since the heat transfer from the bubble to the liquid is

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porportional to the interface area, generally speaking, $\mu_i \neq \mu_j$, $i \neq j$. The specific volume of the mixture can be expressed as

$$v = c_l/\rho_l + \left(4\pi \sum_{i=1}^{M} n_i R_i^3\right)\Big/3, \; c_l + \sum_{i=1}^{M} c_{2i} = 1 \tag{1.5}$$

(c_ℓ is the mass concentration of the liquid). The equation of state of the i-th phase is taken in the form:

$$p_2^i(R_i) = p_0 (R_{i0}/R_i)^{3\gamma} \tag{1.6}$$

(p_0 and R_{i0} are the equilibrium values of the pressure and the bubble radii, $\gamma > 1$ is the polytropic exponent, common to all the bubbles). We will assume that c_{2i} = const, n_i = const.

We seek solutions of the system (1.1)-(1.6) that depend only on the variable $\xi = q - Dt$ (D is the velocity of the traveling wave). The equations for the radii R_i are written as follows:

$$D^2\left(R_i R_i'' + 3R_i'^2/2\right) = \left(p_0 (R_{i0}/R_i)^{3\gamma} - p_0 + D^2 \frac{4}{3}\pi \sum_{i=1}^{M} n_i \left(R_i^3 - R_{i0}^3\right)\right)\Big/\rho_l + \\ + 4\mu_i D R_i'/(\rho_l R_i). \tag{1.7}$$

We introduce the dimensionless variables $x_i = R_i/R_{i0}$, $\tau = \xi/q_0$, where $q_0 > 0$ is a characteristic Lagrangian variable, and the notation

$$\alpha_i = \left(\rho_l D^2 R_{i0}^2\right)/\left(p_0 q_0^2\right), \; \beta_i = (4\mu_i D)/(p_0 q_0), \; \delta_i = \left(4\pi D^2 n_i R_{i0}^3\right)/(3p_0), \tag{1.8}$$

denoting by a dot the derivative with respect to τ. Then (1.7) takes the form:

$$\alpha_i\left(x_i \ddot{x}_i + 3\dot{x}_i^2/2\right) = x_i^{-3\gamma} - 1 + \sum_{j=1}^{M} \delta_j \left(x_j^3 - 1\right) + \beta_i \dot{x}_i/x_i, \; i = 1, \ldots, M. \tag{1.9}$$

Without loss of generality, we may assume that D > 0, since otherwise it is sufficient to make the substitution $\tau \to -\tau$. Thus, $\beta_i > 0$.

By the solution of the system (1.9) we understand the vector function $\mathbf{Z} = (x_1, \ldots, x_M, \dot{x}_1, \ldots, \dot{x}_M) \in \mathbf{R}^{2M}$, $x_i > 0$ to be definite and continuously differentiable for all $\tau \in \mathbf{R}$ and satisfying the boundary conditions

$$\lim_{\tau\to\pm\infty} \mathbf{Z} = \left(x_1^\pm, \ldots, x_M^\pm, 0, \ldots, 0\right) = \mathbf{Z}^\pm.$$

The system (1.9) has the trivial solution $\mathbf{Z}^0 = (1, \ldots, 1, 0, \ldots, 0)$. The problem is to find a nontrivial bounded solution. The case $M \leq 2$ was fully investigated in [4].

2. Transition to Canonical Variables

If the viscosity coefficients $\mu_i = 0$ ($\beta_i = 0$), then the system (1.9) will be Hamiltonian [5]. We now go over to canonial variables. Following [4], we consider the function

$$\widetilde{H}\left(\mathbf{x}, \dot{\mathbf{x}}\right) = \sum_{j=1}^{M} \delta_j \left(\frac{x_j^{3(1-\gamma)} - 1}{\gamma - 1} + \left(x_j^3 - 1\right) + \frac{3}{2}\alpha_j x_j^3 \dot{x}_j^2\right) - \\ - \frac{1}{2}\left(\sum_{j=1}^{M} \delta_j \left(x_j^3 - 1\right)\right)^2, \; \mathbf{x} = (x_1, \ldots, x_M). \tag{2.1}$$

We introduce the notation

$$p_j = \sqrt{3\delta_j \alpha_j}\, x_j^{3/2} \dot{x}_j, \; q_j = (2/5)\sqrt{3\delta_j \alpha_j}\, x_j^{5/2}. \tag{2.2}$$

Then (2.1) can be represented in the form (with the tilde removed from H):

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{M} p_i^2 + V(\mathbf{q}), \tag{2.3}$$

where the potential

$$V(\mathbf{q}) = \tilde{V}(\mathbf{x}(\mathbf{q})) \equiv \sum_{j=1}^{M} \delta_j \left(\frac{x_j^{3(1-\gamma)} - 1}{\gamma - 1} + x_j^3 - 1 \right) - \frac{1}{2} \left(\sum_{j=1}^{M} \delta_j (x_j^3 - 1) \right)^2, \tag{2.4}$$

and $x_j(q_j)$ are determined from (2.2).

Lemma 2.1 [4]. The system of equations (1.9) is equivalent to the following systems:

$$\dot{q}_i = \partial H/\partial p_i, \dot{p}_i = -\partial H/\partial q_i + F_i, \quad F_i = \frac{\beta_i}{\alpha_i} \frac{p_i}{\left(\frac{5 q_i}{2 \sqrt{3 \delta_i \alpha_i}} \right)^{4/5}}, i = 1, \ldots, M. \tag{2.5}$$

It follows from Lemma 2.1 that the derivative of the function H along trajectories of the system (2.5) is nonnegative. This means that a nonconstant bounded solution of the system (2.5), if it exists, must begin and end at singular points.

3. Singular Points and Stability of Traveling Waves

In order to find the singular points we will use the coordinates $\mathbf{x}$, $\dot{\mathbf{x}}$ the system of algebraic equations has the form

$$x_i^{-3\gamma} - 1 + \sum_{j=1}^{M} \delta_j (x_j^3 - 1) = 0. \tag{3.1}$$

We take

$$\delta = \sum_{j=1}^{M} \delta_j. \tag{3.2}$$

From (3.1) it follows [4] that when $\delta = \gamma$ there is only one singular point: $x_{i0} = x_0 = 1$, when $\delta > \gamma$ there are two: $x_{i0} = x_0 = 1$, $x_i^* = x_* < 1$ and when $\delta < \gamma$ there are also two: $x_{i0} = x_0 = 1$, $x_i^* = x^* > 1$, $i = 1, \ldots, M$.

Thus, when $\delta = \gamma$ there are known to be no nonconstant bounded solutions. The cases $\delta < \gamma$ and $\delta > \gamma$ need further investigation. We introduce the notation: $\mathbf{z}_0 = (x_{10}, \ldots, x_{M0}) = (1, \ldots, 1)$, $\mathbf{z}_* = (x_{1*}, \ldots, x_{M*}) = (x_*, \ldots, x_*)$, $\mathbf{z}^* = (x_1^*, \ldots, x_M^*) = (x^*, \ldots, x^*)$.

Lemma 3.1 [4]. If $\delta > \gamma$, then $\tilde{V}(\mathbf{z}_*) < 0 = \tilde{V}(\mathbf{z}_0)$. If $\delta < \gamma$, then $\tilde{V}(\mathbf{z}^*) > 0 = \tilde{V}(\mathbf{z}_0)$.

Since H increases along the trajectories of the system (2.5), it follows from Lemma 3.1 that if $\mathbf{x}(\tau)$ is a nonconstant bounded solution of equations (1.9), then

$$\mathbf{z}_* = \lim_{\tau \to -\infty} \mathbf{x}(\tau), \mathbf{z}_0 = \lim_{\tau \to \infty} \mathbf{x}(\tau) \quad \text{when} \quad \delta > \gamma,$$
$$\mathbf{z}_0 = \lim_{\tau \to -\infty} \mathbf{x}(\tau), \mathbf{z}^* = \lim_{\tau \to \infty} \mathbf{x}(\tau) \quad \text{when} \quad \delta < \gamma.$$

Since $D > 0$, the inequality $\delta > \gamma$ corresponds to the state with a zero index being the state ahead of the wave front, while when $\delta < \gamma$ it is the state behind the front. The physical significance of the inequalities $\delta > \gamma$ and $\delta < \gamma$ will become clear from the following assertion.

Lemma 3.2 [4]. The following chains of equivalent inequalities hold:

$$\delta > \gamma \Leftrightarrow a_e^2(v_0) < v_0^2 D^2 \Leftrightarrow v_*^2 D^2 < a_e^2(v_*) \Leftrightarrow \delta < \gamma x_*^{-3(\gamma+1)},$$
$$\delta < \gamma \Leftrightarrow a_e^2(v_0) > v_0^2 D^2 \Leftrightarrow (v^*)^2 D^2 > a_e^2(v^*) \Leftrightarrow \delta > \gamma (x^*)^{-3(\gamma+1)}.$$

Here, a_e is the equilibrium speed of sound in the mixture:

$$a_e^2(v) = \frac{dp_e(v)}{d(1/v)};\ p_e(v) = p_0\left(\frac{\frac{4}{3}\pi\sum_{i=1}^{M} n_i R_{i0}^3}{v - c_l/\rho_l}\right)^{\gamma};$$

v_0, v_*, and v^* are the values of the specific volume at the corresponding singular points. Thus, Lemma 3.2 is, essentially, the Cemplen theorem for the equilibrium model of a bubbly medium. If the corresponding structure (solution of traveling wave type) exists, then within the framework of the equilibrium model it corresponds to a stable strong discontinuity.

We will investigate only the case $\delta > \gamma$ (x_0 is the state ahead of the wave front), since the inequality $\delta < \gamma$ can be reduced to $\delta > \gamma$ by renotation. In fact, for that it is sufficient when $\delta < \gamma$ to rewrite Eqs. (1.7) in the form:

$$D^2(R_i R_i'' + 3R_i'^2/2) = (p_0(R_{i0}/R_i)^{3\gamma} - p^* + \\ + D^2\frac{4}{3}\pi\sum_{i=1}^{M} n_i(R_i^3 - R_i^{*3}))/\rho_l + 4\mu_i D R_i'/(\rho_l R_i),\quad p^* = p_e(v^*),\quad R_i^* = x^* R_{i0}.$$

If we introduce the dimensionless variables $x_i = R_i/R_i^*$, $\tau = \xi/q_0$ and note that $p^* = p_0(R_{i0}/R_i^*)^{3\gamma}$, then the corresponding equations will have the form of (1.9) with α_i, δ_i, β_i replaced by α_i^*, δ_i^*, β_i^*. In this case the inequality $\delta < \gamma$ goes over into $\delta^* = \sum_{i=1}^{M}\delta_i^* > \gamma$.

Thus, everywhere in what follows only the case $\delta > \gamma$ will be considered.

4. Linearization in the Neighborhood of the Singular Points

Let A^+ and A^- be the linearization matrices of the system (2.5) in the neighborhoods of the points $(\mathbf{q}_0, 0)$, A^- and $(\mathbf{q}_*, 0)$, where $\mathbf{q}_0$ and $\mathbf{q}_*$, which correspond to the points $\mathbf{z}_0 = (1, \ldots, 1)$ and $\mathbf{z}_* = (x_*, \ldots, x_*)$, are recalculated from (2.2).

Lemma 4.1 [4]. We introduce the notation

$$A(x) = \begin{pmatrix} 0 & E \\ \hline T & B \end{pmatrix},$$

where 0 and E are null and unit $M \times M$ matrices, $T = |t_{ij}|_{M\times M}$, and $B = \|b_{ij}\|_{M\times M}$:

$$t_{ij} = \begin{cases} 3x\sqrt{\dfrac{\delta_i\delta_j}{\alpha_i\alpha_j}}, & \text{if } i \neq j, \\ 3x\dfrac{(\delta_i - \gamma x^{-3\gamma-3})}{\alpha_i}, & \text{if } i = j; \end{cases} \tag{4.1}$$

$$b_{ij} = \frac{\beta_i}{\alpha_i}x^{-2}\delta_{ij},\quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.2}$$

Then $A^+ = A(1)$, $A^- = A(x_*)$.

In [4] the properties of the spectrum $A(x)$ were investigated only for $M < 2$. We will carry out a general analysis. Considering the equation $A(x)\langle \mathbf{w}\rangle = \lambda\mathbf{w}$ ($\mathbf{w} \in \mathbf{R}^{2M}$ is a 2M-dimensional vector), we divide $\mathbf{w}$ into two M-dimensional vectors: $\mathbf{w} = (\mathbf{w}', \mathbf{w}'')$. By virtue of the representation of the matrix $A(x)$ we have $\mathbf{w}'' = \lambda\mathbf{w}'$, $T\langle\mathbf{w}'\rangle + (B - \lambda E)\langle\mathbf{w}''\rangle = 0$, from which there follows the equation for the eigenvalues of the matrix $A(x)$

$$\det(T + \lambda B - \lambda^2 E) = 0. \tag{4.3}$$

We introduce the matrices D_1, D_2, W_M in accordance with the following rule:

$$D_1 = \| d_{1ij} \|_{M\times M},\ d_{1ij} = 3x \sqrt{\frac{\delta_i}{\alpha_i}}\,\delta_{ij},\ D_2 = \| d_{2ij} \|_{M\times M},\ d_{2ij} = \sqrt{\frac{\delta_i}{\alpha_i}}\,\delta_{ij},$$

$$W_M = \| w_{Mij} \|_{M\times M},\ w_{Mij} = \begin{cases} 1, & i \neq j, \\ a_i(\lambda), & i = j; \end{cases}$$

$$a_i(\lambda) = \frac{\alpha_i}{3\delta_i x}\left[\lambda\left(\frac{\beta_i}{\alpha_i x^2} - \lambda\right) + \frac{3x}{\alpha_i}\left(\delta_i - \frac{\gamma}{x^{3\gamma+3}}\right)\right]. \tag{4.4}$$

From (4.1), (4.2), and (4.4) it follows that $T + \lambda B - \lambda^2 E = D_1 W_M D_2$, i.e., Eq. (4.3) is equivalent to $\Delta_M \equiv \det W_M = 0$.

Lemma 4.2. Let $M \geq 2$. Then

$$\Delta_M = \det W_M = \prod_{i=1}^{M} (a_i - 1) + \sum_{j=1}^{M} \prod_{\substack{k=1 \\ k\neq j}}^{M} (a_k - 1).$$

Proof. The case M = 2 is obvious. Let the equation be correct for M − 1. We expand the determinant of the matrix W_M with respect to the M-th column. Then

$$\Delta_M = a_M \Delta_{M-1} - \sum_{j=1}^{M-1} \Delta_{M-1}^{j}. \tag{4.5}$$

Here, Δ_{M-1} is the determinant of the matrix W_{M-1}, and Δ_{M-1}^{j} that of the matrix W_{M-1}^{j}, which is organized in the same way as W_{M-1}, except that its j-th row consists entirely of units. By inductive assumption

$$\Delta_{M-1} = \prod_{i=1}^{M-1} (a_i - 1) + \sum_{j=1}^{M-1} \prod_{\substack{k=1 \\ k\neq j}}^{M-1} (a_k - 1),\ \Delta_{M-1}^{j} = \prod_{\substack{k=1 \\ k\neq j}}^{M-1} (a_k - 1). \tag{4.6}$$

The substitution of (4.6) in (4.5) proves the lemma.

Remark. If we set $a_0 = 2$ and

$$\Delta_M = \prod_{i=1}^{M} (a_i - 1) + \sum_{j=1}^{M} \prod_{\substack{i=0 \\ i\neq j}}^{M} (a_i - 1),$$

then the expression for Δ_M will also be correct when M = 1.

We will show that the equation $\Delta_M(\lambda) = 0$ does not have purely imaginary roots. We assume the opposite, i.e., $\lambda = ib$, $b \in \mathbf{R}$ is a root. Since $a_k(ib) - 1 \neq 0$ for all $b \in \mathbf{R}$, the equation $\Delta_M(\lambda) = 0$ is equivalent to $\sum_{j=1}^{M} (a_j(ib) - 1)^{-1} + 1 = 0$. From this there follow the two equations: $b = 0$ and $\delta \equiv \sum_{j=1}^{M} \delta_j = \gamma x^{-3\gamma-3}$. Since $\delta > \gamma \leftrightarrow \delta < \gamma x_*^{-3\gamma-3}$ (Lemma 3.2), the equation $\Delta_M(\lambda) = 0$ does not have roots on the imaginary axis.

We note that $a_j(\lambda) - 1 \neq 0$ when $\lambda \leq 0$. Then when $\lambda \leq 0$ the equation $\Delta_M(\lambda) = 0$ is equivalent to

$$\tilde{\Delta}_M(\lambda) \equiv 1 + \sum_{j=1}^{M} (a_j(\lambda) - 1)^{-1} = 0.$$

By virtue of (4.4) when $\lambda \leq 0$ we find

$$\tilde{\Delta}_M(0) = 1 - \delta x^{3\gamma+3}/\gamma,\ \delta = \sum_{i=1}^{M} \delta_i,\ \frac{d\tilde{\Delta}_M(\lambda)}{d\lambda} < 0,\ \lim_{\lambda \to -\infty} \tilde{\Delta}_M(\lambda) = 1. \tag{4.7}$$

It follows from (4.7) that when x = 1, $\lambda < 0$ the equation $\tilde{\Delta}_M(\lambda) = 0$ has a single root $\lambda_0^- < 0$, and when $x = x_*$, $\lambda < 0$ there are no roots. Since $\Delta_M(\lambda)$ is a polynomial with real coefficients of even degree, when x = 1, $\lambda > 0$ it can be stated that the equation $\Delta_M(\lambda) = 0$ has at least one positive root.

We will show that the matrices A^+ and A^- (see Lemma 4.1) have respectively 2M − 1 and 2M eigenvalues in the right half-plane for all values of α_i, β_i, $\delta_i > 0$. For this it is sufficient to find just a certain interval of the variables α_i, β_i, δ_i on which that is so. In fact, the root of the algebraic equation depend continuously on its coefficients and therefore they can enter the left half-plane only by crossing the imaginary axis from the right. However, as already shown, there are no purely imaginary roots.

Below, we will confine ourselves to the case of the matrix A^+ (A^- can be similarly investigated). We assume that $\alpha_k \neq \alpha_j$, $\beta_k \neq \beta_j$, $k \neq j$, since otherwise all the bubbles are of the same kind. We choose $\beta_k^2 < 12\gamma\alpha_k$, k = 1, ..., M. Then $a_j(\lambda) \neq 1$ for any j = 1, ..., M. Therefore the equation $\Delta_M(\lambda) = 0$ is again equivalent to $\tilde{\Delta}_M(\lambda) = 0$. The latter may conveniently be written in the form:

$$\sum_{k=1}^{M} f_k(\lambda) = 1,\quad f_k(\lambda) = -(a_k(\lambda) - 1)^{-1}, \tag{4.8}$$

$$f_k(\lambda) > 0 \text{ when } \lambda \in R,\ f_k(\lambda) \to 0 \text{ when } \lambda \to \pm\infty.$$

There exists a single maximum of the function $f_k(\lambda)$:

$$\lambda_k^* = \beta_k/(2\alpha_k),\ f_k(\lambda_k^*) = 12\delta_k\alpha_k/(12\gamma\alpha_k - \beta_k^2).$$

We choose $\delta_1 > \lambda$. Then $f_1(\lambda_1^*) > 1$, $f_1(0) > 1$. Therefore the equation $f_1(\lambda) = 1$ has exactly one positive root. Moreover, let $\delta_k \ll \gamma$, k = 2, ..., M. It is possible to choose β_k so that for any k = 2, ..., M

$$3(\gamma - \delta_k)/\alpha_k < \lambda_k^{*2} < 3\gamma/\alpha_k.$$

From this it follows that $f_k(\lambda_k^*) > 1$, k = 2, ..., M. Finally, if we take the α_k (k = 2, ..., M) such that $\alpha_1 \ll \alpha_2 \ll \ldots \ll \alpha_M$, then $\lambda_1^* \ll \lambda_2^* \ll \ldots \ll \lambda_M^*$ and, consequently, the roots of Eq. (4.8) will be similar to the roots of the equations $f_k(\lambda) = 1$ (k = 1, ..., M), each of which, except for $f_1(\lambda) = 1$, has exactly two positive roots. The case $x = x_*$ can be similarly investigated. Thus, we have proved:

Lemma 4.3. For any positive α_i, β_i, γ_i and $\delta > \gamma$ (see (1.8), (3.2)) the matrix A^+ has one negative eigenvalue and 2M − 1 eigenvalues in the right half-plane, and A^- has 2M eigenvalues in the right half-plane.

5. Level Surface of Hamiltonian. Uniqueness Theorem. In what follows we will need to know the structure of the level set of the Hamiltonian $H(\mathbf{q}, \mathbf{p})$, i.e., the set $H_c = \{(\mathbf{q}, \mathbf{p}) | H(\mathbf{q}, \mathbf{p}) = c\}$ (c is a constant). The first step is to find the critical points of the Hamiltonian, which coincide with the singular points of the system (2.5). In view of (2.3) it is sufficient to investigate the potential $V(\mathbf{q})$ whose second-derivative matrix coincides at the singular points with −T (Lemma 4.1). By analogy with (4.3), the equation $\det(-T - \lambda E) = 0$ is equivalent to det G = 0, where $G = \|g_{ij}\|_{M\times M}$,

$$g_{ij} = \begin{cases} \dfrac{\delta_i - \gamma x^{-3\gamma-3}}{\delta_i} + \dfrac{\alpha_i}{3\delta_i x}\lambda, & i = j, \\ 1, & i \neq j. \end{cases}$$

By virtue of Lemma 4.2, the eigenvalues of the matrix $-T$ are roots of the polynomial

$$\prod_{i=1}^{M}(g_{ii}-1)+\sum_{j=1}^{M}\prod_{\substack{i=1\\ i\neq j}}^{M}(g_{ii}-1)=0. \tag{5.1}$$

Since T is a symmetric matrix, all the roots of (5.1) are real. The inequality $\delta > \gamma$ guarantees that when $x = 1$ Eq. (5.1) has only one negative root, and when $x = x_*$ there are no negative roots. All the other roots are positive.

Let V_0 and V_* be the values of the potential $V(\mathbf{q})$ at the points $\mathbf{q}_0$ and $\mathbf{q}_*$ respectively. By virtue of Lemma 3.1 $V_0 = 0$, $V_* < 0$. Let us consider a fairly small neighborhood U_* of the point $(\mathbf{q}_*, \mathbf{0})$ in $\mathbf{R}^{2M}$. In view of the properties of the potential $V(\mathbf{q})$ and Morse's lemma [6, p. 402] there exist regular local coordinates $\mathbf{q}'$ such that in U_*

$$H(\mathbf{q}', \mathbf{p}) = V_* + \frac{1}{2}\left(\sum_{i=1}^{M} p_i^2 + \sum_{i=1}^{M} q_i'^2\right).$$

Therefore the set $H_c = \{(\mathbf{q}, \mathbf{p}) | H(\mathbf{q}, \mathbf{p}) = c\}$ with $(\mathbf{q}, \mathbf{p}) \in U_*$ $(c \in (V_*, V_* + \varepsilon),\ 0 < \varepsilon << 1)$ is diffeomorphic to a sphere S^{2M-1}. Since the interval (V_*, V_0)does not contain a critical level of the function H, for all $c \in (V_*, V_0)$ H_c has a connectivity component diffeomorphic to S^{2M-1}. When $c = 0$ is crossed, the structure of H_c changes. In fact, by virtue of Morse's lemma in the neighborhood U_0 of the point $(\mathbf{q}_0, \mathbf{0})$ there exist regular local coordinates $\mathbf{q}''$ such that

$$H(\mathbf{q}'', \mathbf{p}) = \frac{1}{2}\left(\sum_{i=1}^{M} p_i^2 + \sum_{i=1}^{M-1} q_i''^2 - q_M''^2\right).$$

From this it follows that the set H_0 is locally a cone. In U_0 the surface $H_{-\varepsilon}$ $(0 < \varepsilon << 1)$ has two connectivity components and is diffeomorphic to $S^0 \times E^{2M-1}m$ where E^{2M-1} is a $2M - 1$-dimensional cell (topological image of a $2M - 1$-dimensional disk); the surface H_ε is connected and diffeomorphic in U_0 to $E^1 \times S^{2M-2}$. One of the cells E^{2M-1} corresponds to the intersection of the neighborhood U_0 and the connectivity component of the set $H_{-\varepsilon}$ diffeomorphic to S^{2M-1}, while the second corresponds to the intersection of U_0 and the noncompact connectivity component of the set $H_{-\varepsilon}$.

We introduce the notation $H^- = \{(\mathbf{q}, \mathbf{p}) | H(\mathbf{q}, \mathbf{p}) < 0\}$. From the above analysis it follows that H^- is not connected:

$$H^- = H_*^- \cup \hat{H}^-,\ H_*^- \cap \hat{H}^- = \varnothing. \tag{5.2}$$

Here, the set H_*^- is homeomorphic to an open 2M-dimensional cell and contains the point $(q_*, 0)$, while $\hat{H}^-$ is a noncompact set. This makes it possible to prove the uniqueness theorem:

<u>THEOREM 5.1.</u> If a solution of the system (2.5) connecting the points $(\mathbf{q}_0, \mathbf{0})$ and $(\mathbf{q}_*, \mathbf{0})$ exists, then it is unique.

<u>Proof</u>. By virtue of Lemma 4.3 and the Grobman–Hartman theorem, exactly two trajectories of the system (2.5) arrive at the point $(\mathbf{q}_0, \mathbf{0})$. Let λ_0^- be a negative eigenvalue of the matrix A^+ (Lemma 4.1), and let $\mathbf{r}$ be the corresponding eigenvector. It can be directly verified that

$$(\mathbf{r}, \partial^2 H/\partial \mathbf{x}^2(\mathbf{q}_0, \mathbf{0})\langle \mathbf{r}\rangle) < 0. \tag{5.3}$$

Inequality (5.3) implies that these trajectories lie locally in different connectivity components of the set H^-. We will show that only one of them can begin at the point $(\mathbf{q}_*, 0)$. Let ℓ be a trajectory of the system (2.5) connecting the points $(\mathbf{q}_*, 0)$ and $(\mathbf{q}_0, 0)$. We will show that it always lies within H_*^- (see (5.2)). In fact, if the opposite were true, there would be a moment of time $\tau_* < \infty$ at which ℓ crosses the boundary of the set H_*^-. Since H increases along the trajectories of the system (2.5), when $\tau > \tau^*$ ℓ must lie outside H^-. Consequently, it does not meet $(\mathbf{q}_0, 0)$. Thus, ℓ belongs to H_*^-. Uniqueness is proven.

6. Existence Theorem

Let us consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \mathbf{x} \in \mathbf{R}^n, \mathbf{f}: \mathbf{R}^n \to \mathbf{R}^n, (\dot{\cdot}) = d/dt. \tag{6.1}$$

We denote by $\mathbf{x}\cdot t$ a point on a trajectory of the system (6.1) at the moment t such that $\mathbf{x}\cdot 0 = \mathbf{x}$. Systems of the type (6.1) are associated with the well known concept of an invariant set [7]. THe following definitions and properties [7-9] are given in order to close the exposition.

The set I is called an isolated invariant set if I is a maximal invariant set in some neighborhood of itself. The compact set N is called an isolated neighborhood of the maximal invariant set I contained in N, if I lies strictly within N. The isolating neighborhood B is called an isolating block, if its boundary ∂B can be represented in the form $\partial B = b^+ \cup b^- \cup \tau$, where $b^+(b^-)$ is the set of exit (entrance) points (the trajectories leave (enter) B as time increases), and τ consists of segments of trajectories of the system (6.1) connecting b^+ and b^-.

We will call B/b^+ the quotient space obtained as a result of the contraction of b^+ into a point (pointed set [6]). The class of homotopically equivalent quotient spaces B/b^+ is called the Conley index of the isolated invariant set I. We will assume that $h(I) = [B/b^+]$ (the square brackets denote homotopytype). In [8, 9] it was shown that h(I) does not depend on the choice of isolating block. The indices can be added. If I_1 and I_2 are nonintersecting isolated invariant sets, then $h(I_1 \cup I_2) = h(I_1) \vee h(I_2)$ ($\vee$ denotes the bouquet of corresponding pointed spaces [6]). The Conley index is "nonnegative," i.e., if $h(I_1) \vee h(I_2) = \bar{0}$, where $\bar{0}$ is the homotopy type of the contracted space (homotopically equivalent to a point), then $h(I_1) = h(I_2) = \bar{0}$.

Example [8, 9]. Let x_0 be a rest point of (6.1), and A_0 the linearization matrix of (6.1) at the point x_0, which does not have eigenvalues on the imaginary axis. Then x_0 is an isolated invariant set, and $h(x_0) = \Sigma^k$ is a pointed sphere of dimensionality k (k is the number of eigenvalues of A_0 lying in the right half-plane).

By virtue of Lemma 4.3 $h(\{\mathbf{q}_0, 0\}) = \Sigma^{2M-1}$, $h(\{\mathbf{q}_*, 0\}) = \Sigma^{2M}$.

THEOREM 6.1. There exists a trajectory of the system (2.5) connecting the rest points $(\mathbf{q}_*, 0)$ and $(\mathbf{q}_0, 0)$.

Proof. We will construct the isolating block B. Let $\mathbf{r}$ be an eigenvector of the matrix A^+ corresponding to a negative eigenvalue. In the small neighborhood U_0 of the point $(\mathbf{q}_0, 0)$ we will consider a family Λ of hyperplanes orthogonal to $\mathbf{r}$ such that the points $(\mathbf{q}_*, 0)$ and $(\mathbf{q}_0, 0)$ lie on the same side of Λ. We fix $0 < \varepsilon << 1$. We select the hyperplane $L \in \Lambda$ intersecting $H_{-\varepsilon} \cap \hat{H}^-$ in U_0 in a set homeomorphic to S^{2M-2}. Such an L exists by virtue of the properties of the surface $H_{-\varepsilon}$ (see Sec. 5). By virtue of (5.3) the part of the hypersurface $H_{-\varepsilon} \cap \hat{H}^-$ located with the singular points on the same side of L, which is a 2M − 1-dimensional cell, will be the entrance set b^-. We release from points on the boundary of the set b^- the trajectories of the system (2.5) that form the set τ in the definition of the block B and intersect H_ε in a set of homeomorphic to S^{2M-2}. The sphere S^{2M-2} divides H_ε into two nonintersecting parts: $H_\varepsilon = H_{*\varepsilon} \cup \hat{H}_\varepsilon$, where $H_{*\varepsilon}$ is homeomorphic to the hemisphere S^{2M-1}, and $\hat{H}_\varepsilon$ is the noncompact part. Here $H_{*\varepsilon}$ is the set of exit points b^+. The block B has been constructed. By virtue of the fact that b^+ is simply connected the quotient space B/b^+ is a 2M-dimensional cell. Consequently, the homotopy type $[B/b^+] = \bar{0}$.

On the other hand, if no trajectory connecting the points $(\mathbf{q}_*, 0)$ and $(\mathbf{q}_0, 0)$ exists, then the only invariant sets of the system (2.5) are the rest points themselves. Consequently, using the Conley index summation rule, we have $\Sigma^{2M-1} \vee \Sigma^{2M} = \bar{0}$, which contradicts the "nonnegativity" of the index. QED.

LITERATURE CITED

1. S. V. Iordanskii, "Equations of motion of a liquid containing gas bubbles," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1960).
2. B. S. Kogarko, "One-dimensional unsteady motion of a liquid with onset and development of cavitation," Dokl. Akad. Nauk SSSR, 155, No. 4 (1964).
3. R. I. Nigmatulin, Dynamics of Multiphase Media [in Russian], Parts 1, 2, Nauka, Moscow (1987).
4. S. L. Gavrilyuk, "Existence, uniqueness and Lax stability of traveling waves in a polydisperse bubbly medium with dissipation," in: Partial Differential Equations. Collection [in Russian], IM USSR Acad. Sci. Sib. Branch (1989).
5. V. G. Gasenko and V. L. Izergin, "Nonlinear wave stochastization in a polydisperse gas-liquid mixture," Proc. 11th Intern. Symp. on Nonlinear Acoustics, Part 2, Novosibirsk (1987).
6. A. T. Fomenko and D. B. Fuks, Course of Homotopic Topology [in Russian], Nauka, Moscow (1989).
7. Dynamical Systems – 1, in: Advances in Science and Engineering. Ser. Modern Problems of Mathematics. Fundamental Trends [in Russian], VINITI, Moscow (1985).
8. J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York (1983).
9. C. Conley, Isolated Invariant Sets and the Morse Index, Am. Math. Soc., Providence (1978). (Conf. Board Math. Sci., No. 38).